KNOTTED HANDLE DECOMPOSING SPHERES FOR HANDLEBODY-KNOTS

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ABSTRACT. We show that a handlebody-knot whose exterior is boundary-irreducible has a unique maximal unnested set of knotted handle decomposing spheres up to isotopies and annulus-moves. As an application, we show that the handlebody-knots 6_{14} and 6_{15} are not equivalent. We also show that some genus two handlebody-knots with a knotted handle decomposing sphere can be determined by their exteriors. As an application, we show that the exteriors of 6_{14} and 6_{15} are not homeomorphic.

1. INTRODUCTION

A *genus g handlebody-knot* is a genus g handlebody embedded in the 3-sphere S^3. Two handlebody-knots are *equivalent* if one can be transformed into the other by an isotopy of S^3. A handlebody-knot is *trivial* if it is equivalent to a handlebody standardly embedded in S^3, whose exterior is a handlebody. We denote by $E(H) = S^3 - \text{int } H$ the exterior of a handlebody-knot H.

Definition 1.1. A 2-sphere S in S^3 is an *n-decomposing sphere* for a handlebody-knot H if

(1) $S \cap H$ consists of n essential disks in H, and
(2) $S \cap E(H)$ is an incompressible and not boundary-parallel surface in $E(H)$.

In some cases it might be suitable to replace the condition (2) in Definition 1.1 with the condition

(2)' $S \cap E(H)$ is an incompressible, boundary-incompressible, and not boundary-parallel surface in $E(H)$,

although we adopt the condition (2) in this paper. The two definitions are equivalent if $n = 1$, or $n = 2$ and $E(H)$ is boundary-irreducible.

A handlebody-knot H is *reducible* if there exists a 1-decomposing sphere for H, where we remark that (2) follows from (1) when $n = 1$. A handlebody-knot is *irreducible* if it is not reducible. A handlebody-knot H is irreducible if $E(H)$ is boundary-irreducible. The converse is true for a genus two handlebody-knot H. In particular, for a genus two handlebody-knot H, the following are equivalent:

(1) H is irreducible.
(2) $\pi_1(E(H))$ is indecomposable with respect to free products.
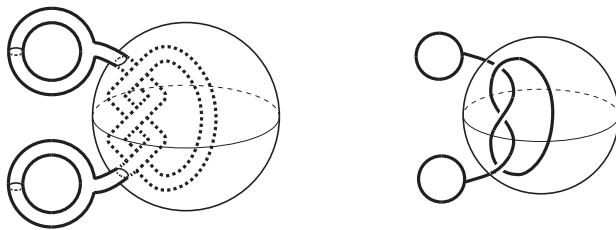(3) $E(H)$ is boundary-prime.
(4) $E(H)$ is boundary-irreducible.

By [16], we have the equivalence between (1) and (2). By [7], we have the equivalence between (2) and (3) for a handlebody-knot H of arbitrary genus. The conditions (3) and (4) are equivalent if $E(H)$ is not a solid torus (cf. [14, Proposition 2.15]). We remark that there is an irreducible genus $g \neq 2$ handlebody-knot whose exterior is not boundary-irreducible (cf. [14, Theorem 5.4]).

The decomposition by 1-decomposing spheres is unique for a trivial handlebody-knot and a genus two handlebody-knot [15]. The uniqueness is not known for a genus $g \geq 3$ handlebody-knot.

Definition 1.2. A 2-sphere S in S^3 is a *knotted handle decomposing sphere* for a handlebody-knot H if

    (1) $S \cap H$ consists of two parallel essential disks in H, and
    (2) $S \cap E(H)$ is an incompressible and not boundary-parallel surface in $E(H)$.

We say that a 2-sphere S bounds $(B, K; H)$ if S bounds a 3-ball B so that $S \cap H$ consists of two parallel essential disks in H, and that $H \cup E(B)$ is equivalent to a regular neighborhood of a nontrivial knot K. A knotted handle decomposing sphere for H bounds $(B, K; H)$. A 2-sphere S which bounds $(B, K; H)$ is not always a knotted handle decomposing sphere for H (see the left picture of Figure 1). In this paper, we represent a handlebody-knot by a spatial trivalent graph whose regular neighborhood is the handlebody-knot as shown in Figure 1. Then the intersection of the spatial trivalent graph and the 2-sphere indicates two disks.

If H is a genus $g \geq 2$ handlebody-knot whose exterior is boundary-irreducible, then a 2-sphere S which bounds $(B, K; H)$ is a knotted handle decomposing sphere for H, where we note that $g \geq 2$ implies that $S \cap E(H)$ is not boundary-parallel in $E(H)$, and that the boundary-irreducibility implies the incompressibility of $S \cap E(H)$. A trivial handlebody-knot has no knotted handle decomposing sphere, since the decomposition by 1-decomposing spheres is unique for a trivial handlebody-knot.

In [6], Moriuchi, Suzuki and the first and second authors gave a table of genus two handlebody-knots up to six crossings, and classified them according to the crossing number and the irreducibility. There are three pairs of handlebody-knots whose fundamental groups are isomorphic in the table. S. Lee and J. H. Lee [10] gave inequivalent genus two handlebody-knots with homeomorphic exteriors including the two pairs 5_1, 6_4 and 5_2, 6_{13} in the table, and distinguish them by classifying essential surfaces in the exteriors. We note that Motto [12] gave different examples with homeomorphic exteriors which do not appear in the above table.

The pair 6_{14}, 6_{15} is the remaining pair of handlebody-knots whose fundamental groups are isomorphic. In Section 2, we show that a handlebody-knot whose exterior is boundary-irreducible has a unique maximal unnested set of knotted handle decomposing spheres up to isotopies and annulus-moves. As an application, we show that the handlebody-knots 6_{14} and 6_{15} are not equivalent. In Section 3, we show that some genus two handlebody-knots with a knotted handle decomposing sphere can be determined by their exteriors. As an application, we show that the exteriors of the handlebody-knots 6_{14} and 6_{15} are not homeomorphic.

2. A UNIQUE DECOMPOSITION FOR A HANDLEBODY-KNOT

Let H be a handlebody-knot in S^3, and S a knotted handle decomposing sphere for H which bounds $(B, K; H)$. Let A be an annulus properly embedded in $E(H) -$ int B so that $A \cap S = l$ is an essential loop in the annulus $S \cap E(H)$, and that $A \cap \partial H = l'$ bounds an essential disk D in H, where $\partial A = l \cup l'$ (see Figure 2). Put $T = (S \cap E(H)) \cup (B \cap \partial H)$. Let A' be an annulus obtained from T by cutting along l and pasting two copies of A, where T is slightly isotoped so that $T \cap H = \emptyset$. Then we have a new knotted handle decomposing sphere S' obtained from A' by attaching two parallel copies of D to $\partial A'$. We say that S' is obtained from S by an *annulus-move* along A. For example, in Figure 3, S' is obtained from S by an annulus-move along A, where we note that S and S' are not isotopic in the exterior of the handlebody-knot.
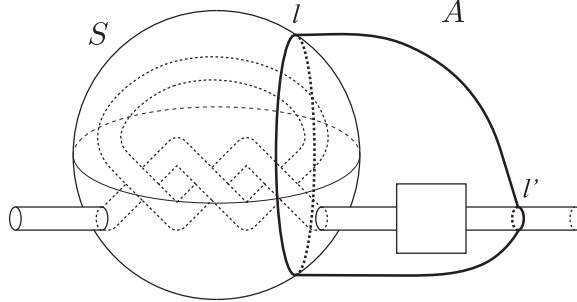

FIGURE 2. An annulus-move along A

A set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of knotted handle decomposing spheres for a handlebody-knot H is *unnested* if each sphere S_i bounds $(B_i, K_i; H)$ so that $B_i \cap B_j = \emptyset$ for $i \neq j$. An unnested set $\mathcal{S}$ is *maximal* if $n \geq m$ for any unnested set $\{S'_1, \ldots, S'_m\}$ of knotted handle decomposing spheres for H. By the Haken–Kneser finiteness theorem [4, 8], there exists a maximal unnested set of knotted handle decomposing spheres for H. By Schubert's theorem [13], K_i is prime for any i if $\mathcal{S}$ is maximal.

Lemma 2.1. *Let H be a handlebody-knot whose exterior is boundary-irreducible. Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ be an unnested set of knotted handle decomposing spheres for H such that S_i bounds $(B_i, K_i; H)$ and that K_i is prime for any i. Let $\mathcal{S}' = \{S'_1, \ldots, S'_m\}$ be a set of 2-decomposing spheres for H. Then $\mathcal{S}$ can be deformed so that $S_i \cap S'_j = \emptyset$ for any i, j by isotopies and annulus-moves.*
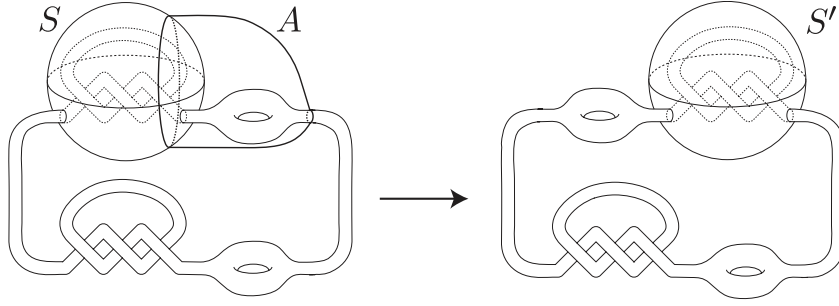
FIGURE 3.

Proof. Put $A_i = S_i \cap E(H)$ for $i = 1, \ldots, n$ and $A'_j = S'_j \cap E(H)$ for $j = 1, \ldots, m$. We may assume that $A_i \cap A'_j$ consists of essential arcs or loops in both A_i and A'_j, and that $|A_i \cap A'_j|$ is minimal up to isotopies and annulus-moves.

In the case that $A_i \cap A'_j$ consists of essential arcs for some i and j, let Δ be a component of $A'_j \cap B_i$ which is cobounded by two adjacent arcs of $A_i \cap A'_j$ in A'_j. By the minimality of $|A_i \cap A'_j|$, $\partial \Delta$ winds around $B_i - \text{int } H$ longitudinally twice, where we note that $B_i - \text{int } H$ is the exterior of K_i. Hence Δ is an essential disk in $B_i - \text{int } H$, which implies that $B_i - \text{int } H$ is boundary-reducible and K_i is a trivial knot, a contradiction.

In the case that $A_i \cap A'_j$ consists of essential loops for some i and j, let F be an outermost subannulus of A'_j which is cut by $A_i \cap A'_j$. If F is contained in B_i, then by the primeness of K_i, we can isotope off F from B_i. Hence F is in the outside of B_i. Then by an annulus move for S_i along the annulus F, we can reduce $|A_i \cap A'_j|$. This contradicts to the minimality of $|A_i \cap A'_j|$. $\square$

Theorem 2.2. *A handlebody-knot H whose exterior is boundary-irreducible has a unique maximal unnested set of knotted handle decomposing spheres for H up to isotopies and annulus-moves.*

Proof. Let $\mathcal{S} = \{S_1, \ldots, S_n\}$, $\mathcal{S}' = \{S'_1 \ldots, S'_n\}$ be maximal unnested sets of knotted handle decomposing spheres for H such that S_i and S'_j bound $(B_i, K_i; H)$ and $(B'_j, K'_j; H)$, respectively. By Lemma 2.1, we can deform $\mathcal{S}'$ so that $S_i \cap S'_j = \emptyset$ for any i, j by isotopies and annulus-moves. We also deform $\mathcal{S}'$ so that $B_i \cap B'_j = \emptyset$ by isotopies if $B_i \cap B'_j$ is homeomorphic to $S^2 \times I$, where I is an interval. Then we have $B_i \subset B'_j$, $B'_j \subset B_i$, or $B_i \cap B'_j = \emptyset$ for any i, j. Since $\mathcal{S}'$ is maximal, for any B_i, there exists a 3-ball B'_j such that $B_i \subset B'_j$ or $B'_j \subset B_i$. Since K_i and K'_j are prime, S_i is parallel to S'_j. This gives a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{S}'$. Hence a maximal unnested set of knotted handle decomposing spheres for H is unique up to isotopies and annulus-moves. $\square$

Proposition 2.3. *Let H be a genus g handlebody-knot whose exterior is boundary-irreducible. Let $\{S_1, \ldots, S_n\}$ be an unnested set of knotted handle decomposing spheres for H such that S_i bounds $(B_i, K_i; H)$ for any i. Put $H' := H \cup B_{m+1} \cup \cdots \cup B_n$. Then $\{S_1, \ldots, S_m\}$ is an unnested set of knotted handle decomposing spheres for H', or $g = 1$ and $m = 1$.*

Proof. Suppose that $S_i \in \{S_1, \ldots, S_m\}$ is not a knotted handle decomposing sphere for H'. If $S_i \cap E(H')$ is compressible in $E(H')$, then $S_i \cap E(H)$ is also compressible in $E(H)$, a contradiction. If $S_i \cap E(H')$ is parallel to an annulus $A \subset \partial E(H')$ in $E(H')$, then A contains some annuli of $(B_{m+1} \cup \cdots \cup B_n) \cap \partial H'$. This shows that $g = 1$ and $m = 1$. $\qquad\square$

Proposition 2.4. *Let H be a genus $g \geq 2$ handlebody-knot, S a 2-sphere which bounds $(B, K; H)$. If $E(H \cup B)$ is boundary-irreducible, then so is $E(H)$.*

Proof. Suppose that $E(H)$ is boundary-reducible and let D be a compressing disk in $E(H)$. Since $E(H \cup B)$ is boundary-irreducible, D intersects with the annulus $A = S \cap E(H)$. Since $E(H)$ is irreducible, we may assume that $D \cap A$ consists of essential arcs in A. Since the knot K is nontrivial, an outermost disk of D gives a compressing disk in $E(H \cup B)$. This is a contradiction. $\qquad\square$

A *(n-component) handlebody-link* is a disjoint union of n handlebodies embedded in the 3-sphere S^3. A *non-split handlebody-link* is a handlebody-link H which has no disjoint 3-balls B_1, B_2 such that $H \cap B_1 \neq \emptyset$, $H \cap B_2 \neq \emptyset$.

Proposition 2.5. *Let H be a handlebody-knot, S a 2-sphere which bounds $(B, K; H)$. Suppose that $H - \operatorname{int} B$ is a non-split handlebody-link whose exterior is boundary-irreducible. If $H - \operatorname{int} B$ is 2-component handlebody-link or $E(H \cup B)$ is a handlebody, then $E(H)$ is boundary-irreducible.*

Proof. Suppose that $E(H)$ is boundary-reducible. Let D be a compressing disk in $E(H)$. Put $A = S \cap E(H)$. If $D \cap A \neq \emptyset$, then we may assume that $D \cap A$ consists of essential arcs in A, since $E(H)$ is irreducible. Since the knot K is nontrivial, an outermost disk δ of D is contained in $E(H \cup B)$. If $H - \operatorname{int} B$ is not a handlebody-knot, then the arc $\delta \cap (H - \operatorname{int} B)$ connects the different components of $H - \operatorname{int} B$ on $\partial(H - \operatorname{int} B)$, a contradiction. If $E(H \cup B)$ is a handlebody, then δ cuts $E(H \cup B)$ into $E(H - \operatorname{int} B)$, which is a handlebody. This implies that $H - \operatorname{int} B$ is trivial, which contradicts that $E(H - \operatorname{int} B)$ is boundary-irreducible. Then $D \cap A = \emptyset$, and so D is in $E(H - \operatorname{int} B)$. Since $E(H - \operatorname{int} B)$ is boundary-irreducible, D is inessential in $E(H - \operatorname{int} B)$. Let D' be a disk in $\partial E(H - \operatorname{int} B)$ such that $\partial D' = \partial D$.

Let D_1, D_2 be the disks such that $S \cap H = D_1 \cup D_2$. If $D' \cap (D_1 \cup D_2) = \emptyset$, then $\partial D'$ is inessential in $\partial E(H)$, which contradicts that D is essential in $E(H)$. If $D' \cap (D_1 \cup D_2) = D_1$ or $D' \cap (D_1 \cup D_2) = D_2$, then the 2-sphere $S' = D' \cup D$ can be slightly isotoped so that $S' \cap (H - \operatorname{int} B) = \emptyset$, which contradicts that $H - \operatorname{int} B$ is non-split, since S' separates D_1 and D_2. Thus $D_1, D_2 \subset D'$. If $H - \operatorname{int} B$ is not a handlebody-knot, then D' connects the different components of $H - \operatorname{int} B$ on $\partial(H - \operatorname{int} B)$, a contradiction. If $E(H \cup B)$ is a handlebody, then the 2-sphere $S' = D' \cup D$ can be slightly isotoped so that D' is properly embedded in $H - \operatorname{int} B$. Then S' separates a handlebody $E(H \cup B)$ into a solid torus and a handlebody which is the exterior of $H - \operatorname{int} B$. This contradicts that $H - \operatorname{int} B$ is nontrivial. $\quad\square$

Example 2.6. We show that any two of the handlebody-knots 5_4, 5_4^*, 6_{14}, 6_{14}^*, 6_{15} and 6_{15}^* are not equivalent, where 5_4, 6_{14} and 6_{15} are the handlebody-knots depicted in Figure 4, and 5_4^*, 6_{14}^* and 6_{15}^* are their mirror images, respectively.

Let H be one of the handlebody-knots 5_4, 5_4^*, 6_{14}, 6_{14}^*, 6_{15} and 6_{15}^*. Let S be the knotted handle decomposing sphere for H depicted in Figure 4, where S bounds $(B, K; H)$ and K is a trefoil knot. By Proposition 2.5, $E(H)$ is boundary-irreducible. By Proposition 2.3, $\{S\}$ is a maximal unnested set of knotted handle
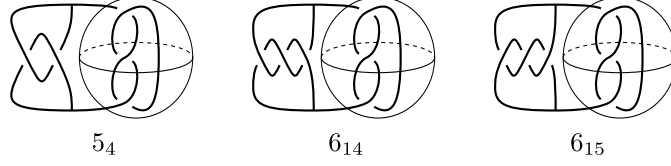
5_4 $\qquad$ 6_{14} $\qquad$ 6_{15}

FIGURE 4.

decomposing spheres for H, since the trivial handlebody-knot $H \cup B$ has no knotted handle decomposing sphere. Then S is unique by Theorem 2.2, which implies that the pair $(K, H - \text{int } B)$ is an invariant of H. Hence any two of the handlebody-knots 5_4, 5_4^*, 6_{14}, 6_{14}^*, 6_{15} and 6_{15}^* are not equivalent.

Proposition 2.7. *There exists a sequence of handlebody-knots H_n $(n \in \mathbb{N} \cup \{0\})$ satisfying the following conditions.*

- *H_0 is the trivial genus two handlebody-knot, which has no knotted handle decomposing sphere.*
- *H_n has a unique knotted handle decomposing sphere S_n which bounds $(B_n, K_n; H_n)$ for $n \geq 1$.*
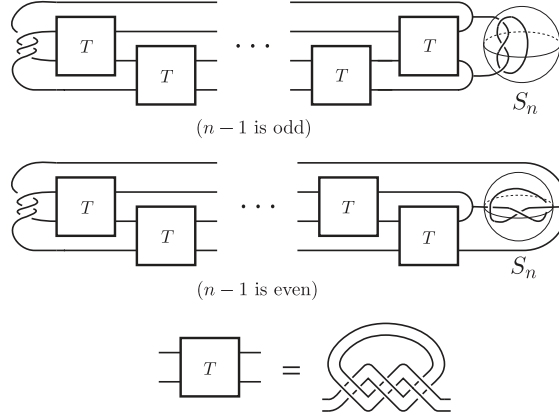- *$H_n \cup B_n$ is equivalent to H_{n-1} as a handlebody-knot.*

FIGURE 5.

Proof. Let H_0 be the trivial genus two handlebody-knot. For $n \geq 1$, let H_n be the genus two handlebody-knot with $n - 1$ tangles T and a 2-sphere S_n bounding $(B_n, K_n; H_n)$ as depicted in Figure 5. Then $H_n \cup B_n$ is equivalent to H_{n-1}. We remark that H_1 is the irreducible handlebody-knot 6_{14}. It follows by Proposition 2.4 that H_n is irreducible for $n \geq 1$. Then S_n is a knotted handle decomposing sphere for H_n.

We prove by induction on n that S_n is a unique knotted handle decomposing sphere for H_n. We already showed that S_1 is a unique knotted handle decomposing sphere for H_1 in Example 2.6. Assume that S_{n-1} is a unique knotted handle decomposing sphere for H_{n-1}. Suppose that S_n is not a unique knotted handle

decomposing sphere for H_n. Then, by Lemma 2.1 and Theorem 2.2, there is a knotted handle decomposing sphere S'_n for H_n which bounds $(B'_n, K'_n; H_n)$ such that the set $\{S_n, S'_n\}$ is a maximal unnested set of knotted handle decomposing spheres for H_n.

Let K^-_n be the core of $H_n - \mathrm{int}\, B_n$, which is a satellite knot. Let T' be the tangle obtained from T and 3 half twists as the leftmost tangle of K^-_n in Figure 5. Then T and T' are prime tangles (cf. [5]). Since K^-_n is obtained from T' and $n-2$ copies of T by tangle sum, K^-_n is a prime knot [11]. It follows by Proposition 2.3 that S'_n corresponds to S_{n-1}. Hence K'_n is the positive trefoil knot, and $(H_n \cup B_n) - \mathrm{int}\, B'_n$ is a regular neighborhood of K^-_{n-1}. A loop l of $S'_n \cap \partial H_n$ is in $\partial(H_n - \mathrm{int}\, B_n)$, since the set $\{S_n, S'_n\}$ is unnested.

If l is essential in $\partial(H_n - \mathrm{int}\, B_n)$, then l is a meridian loop of a solid torus $H_n - \mathrm{int}\, B_n$. By the primeness of K^-_n, the positive trefoil knot K'_n is equivalent to the satellite knot K^-_n for $n > 1$, a contradiction.

If l is inessential in $\partial(H_n - \mathrm{int}\, B_n)$, then l bounds a disk D in $\partial(H_n - \mathrm{int}\, B_n)$. Let D_1, D_2 be the disks such that $S_n \cap H_n = D_1 \cup D_2$. Since l is essential in ∂H_n, $D \cap (D_1 \cup D_2) \neq \emptyset$. If D contains both D_1 and D_2, then l is a separating loop in ∂H_n and ∂H_{n-1}, which contradicts that $S_{n-1} \cap \partial H_{n-1}$ consists of non-separating disks. Thus D contains either D_1 or D_2, which implies that l is parallel to the loops of $S_n \cap \partial H_n$. Then $H_n - \mathrm{int}\, B_n$ and $(H_n \cup B_n) - \mathrm{int}\, B'_n$ are equivalent as handlebody-knots. It follows that K^-_n and K^-_{n-1} are equivalent, which contradicts that K^-_m has a non-trivial Fox 3-coloring if and only if m is odd, since the replacement of the tangle T with the trivial tangle does not change the number of Fox 3-colorings.

Therefore S_n is a unique knotted handle decomposing sphere for H_n. This completes the proof. □

3. Handlebody-knots and their exteriors

In this section, we show that some genus two handlebody-knots with a knotted handle decomposing sphere can be determined by their exteriors. As an application, we show that the exteriors of the handlebody-knots 6_{14} and 6_{15} are not homeomorphic.

Theorem 3.1. *For $i = 1, 2$, let H_i be an irreducible genus two handlebody-knot with a knotted handle decomposing sphere S_i bounding $(B_i, K_i; H_i)$ such that B_i contains all spheres in a maximal unnested set of knotted handle decomposing spheres for H_i. Suppose that $E(H_i \cup B_i)$ is a handlebody and that $H_i - \mathrm{int}\, B_i$ is a nontrivial handlebody-knot for $i = 1, 2$. Then H_1 and H_2 are equivalent if and only if there is an orientation preserving homeomorphism from $E(H_1)$ to $E(H_2)$.*

An annulus is *essential* if A is incompressible and not boundary-parallel. To prove Theorem 3.1, we give some lemmas.

Lemma 3.2 ([2, 15.26 Lemma])**.** *Let K be a knot in S^3. If $E(K)$ contains a essential annulus A, then either*

  (1) *K is a composite knot and A can be extended to a decomposing sphere for K,*
  (2) *K is a torus knot and A can be extended to an unknotted torus or*
  (3) *K is a cable knot and A is the cabling annulus.*

Lemma 3.3 ([9, Lemma 3.2])**.** *If A is a essential annulus in a genus two handlebody W, then either*

(1) A cuts W into a solid torus W_1 and a genus two handlebody W_2 and there is a complete system of meridian disks $\{D_1, D_2\}$ of W_2 such that $D_1 \cap A = \emptyset$ and $D_2 \cap A$ is an essential arc in A, or

(2) A cuts W into a genus two handlebody W' and there is a complete system of meridian disks $\{D_1, D_2\}$ of W' such that $D_1 \cap A$ is an essential arc in A.

We say that an annulus A is obtained from a knotted handle decomposing sphere S for a handlebody-knot H when $A = S \cap E(H)$.

Lemma 3.4. *Let H be an irreducible genus two handlebody-knot with a knotted handle decomposing sphere S bounding $(B, K; H)$ such that B contains all spheres in a maximal unnested set of knotted handle decomposing spheres for H. Suppose that $E(H \cup B)$ is a handlebody and that $H - \operatorname{int} B$ is a nontrivial handlebody-knot. Then any essential separating annulus in $E(H)$ is isotopic to either a cabling annulus for $H - \operatorname{int} B$ or an annulus obtained from a knotted handle decomposing sphere for H.*

Proof. Let A' be a essential separating annulus in $E(H)$. Assuming that A' cannot be obtained from a knotted handle decomposing sphere for H, we show that A' is a cabling annulus for $H - \operatorname{int} B$. Put $A = S \cap E(H)$ and $W = E(H \cup B)$. We may assume that $A \cap A'$ consists of essential arcs or loops in both A and A', and that $|A \cap A'|$ is minimal up to isotopy. As the proof of Lemma 2.1, we may assume that $A \cap A'$ consists of essential loops.

If $\partial A'$ is contained in B, then A' is an annulus obtained from a knotted handle decomposing sphere for H, since each loop of $\partial A'$ is parallel to $\partial(S \cap H)$. Hence there is a loop C of $\partial A'$ contained in W.

Suppose $A \cap A' \neq \emptyset$. Let F be the outermost subannulus on A' containing C, which is an annulus properly embedded in W. Since A' is incompressible in $E(H)$, F is incompressible in W. By the minimality of $|A \cap A'|$, F is not boundary-parallel in W. Let D be a disk in $E(H - \operatorname{int} B)$ such that $D \cap W = F$ and $D \cap B$ is a disk D_0 in B. If C is essential in $\partial(H - \operatorname{int} B)$, then $E(H - \operatorname{int} B)$ is boundary-reducible, which implies that $H - \operatorname{int} B$ is trivial, a contradiction. Hence C is inessential in $\partial(H - \operatorname{int} B)$. Let D' be the disk in $\partial(H - \operatorname{int} B)$ such that $\partial D' = C$. Let D_1, D_2 be the disks such that $S \cap H = D_1 \cup D_2$. If C is parallel to ∂D_0 on $\partial(H \cup B)$, then F is an annulus obtained from a knotted handle decomposing sphere for the trivial genus two handlebody-knot $H \cup B$, a contradiction. Thus $D_1, D_2 \subset D'$ or $(D_1 \cup D_2) \cap D' = \emptyset$, which contradicts that the 2-sphere $S' = D' \cup D$ separates D_1 and D_2, where S' is slightly isotoped so that D' is properly embedded in $H - \operatorname{int} B$. Hence $A \cap A' = \emptyset$, which implies that $A' \subset W$.

The annulus A' is incompressible in W, since it is incompressible in $E(H)$. If A' is boundary-parallel in W, then A' is parallel to A and is obtained from a knotted handle decomposing sphere for H, since A' is not boundary-parallel in $E(H)$. Hence A' is essential in the genus two handlebody W.

By Lemma 3.3, the separating annulus A' cuts W into a solid torus W_1 and a genus two handlebody W_2 so that A' winds around W_1 at least twice. If A is contained in $\partial W \cap W_1$, then by attaching a 2-handle $N(D)$ to the solid torus W_1, we have a once punctured lens space $L(p, q)$ $(p \geq 2)$, where D is a component of $S \cap H$. This contradicts Alexander's theorem [1]. Thus A is contained in $\partial W \cap W_2$ and A' cuts $W \cup B$ into W_1 and $W_2 \cup B$.

Suppose that A' is compressible in $W \cup B$. Let D be a compressing disk for A' in $W \cup B$. Then D is contained in $W_2 \cup B$, since A' is incompressible in W. By attaching a 2-handle $N(D)$ to the solid torus W_1, we have a once punctured lens space $L(p, q)$ $(p \geq 2)$, a contradiction. Thus A' is incompressible in $W \cup B$. Suppose that A' is boundary-parallel in $W \cup B$. Since A' is not boundary-parallel in W, $W_2 \cup B$ is a solid torus $A' \times I$. Then the solid torus W_1 is isotopic to $W \cup B = E(H - \text{int } B)$, which implies that $H - \text{int } B$ is trivial, a contradiction. Thus A' is not boundary-parallel in $W \cup B$. Therefore A' is essential in $W \cup B = E(H - \text{int } B)$, which is the exterior of the tunnel number one knot represented by the core curve of $H - \text{int } B$. By Lemma 3.2, A' is a cabling annulus for $H - \text{int } B$, where we note that a tunnel number one knot is prime. $\square$

Proof of Theorem 3.1. If H_1 and H_2 are equivalent, then there is an orientation preserving self-homeomorphism of S^3 which sends H_1 to H_2, which gives an orientation preserving homeomorphism from $E(H_1)$ to $E(H_2)$.

Suppose that there is an orientation preserving homeomorphism f from $E(H_1)$ to $E(H_2)$. Since any cabling annulus cuts off a solid torus from $E(H_2)$, it follows from Lemma 3.4 that $f(S_1 \cap E(H_1)) = S_2 \cap E(H_2)$. Since $E(H_i - \text{int } B_i)$ and $B_i - \text{int } H_i$ are exteriors of knots, by the Gordon-Luecke theorem [3], both of the restrictions of f to $E(H_1 - \text{int } B_1)$ and $B_1 - \text{int } H_1$ are extended to homeomorphisms of S^3. Hence f can be extended to a homeomorphism $\hat{f}$ of S^3 such that $\hat{f}(S_1) = S_2$ and $\hat{f}(H_1) = H_2$. $\square$

Example 3.5. By Example 2.6, neither 6_{15} nor 6_{15}^* is equivalent to 6_{14}. We recall that each of them has a unique knotted handle decomposing sphere. By Theorem 3.1, there is no orientation preserving/reversing homeomorphism from $E(6_{14})$ to $E(6_{15})$. Hence $E(6_{14})$ and $E(6_{15})$ are not homeomorphic.

References

1. J. W. Alexander, *On the subdivision of* 3-*space by a polyhedron*, Proc. Nat. Acad. Sci. USA **10** (1924), 6–8.
2. G. Burde and H. Zieschang, *Knots*, de Gruyter Studies in Mathematics **5**, 1985.
3. C. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), 371–415.
4. W. Haken, *Some Results on Surfaces in* 3-*Manifolds*, Studies in Modern Topology pp. 39–98 Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.), 1968.
5. C. Hayashi, H. Matsuda and M. Ozawa, *Tangle decompositions of satellite knots*, Revista. Mathematica Complutense **12** (1999), 417–437.
6. A. Ishii, K. Kishimoto, H. Moriuchi and M. Suzuki, *A table of genus two handlebody-knots up to six crossings*, J. Knot Theory Ramifications **21** (2012), 1250035, 9 pp.
7. W. Jaco, *Three-manifolds with fundamental group a free product*, Bull. Amer. Math. Soc. **75** (1969), 972–977.
8. H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jahresber. Deutsch. Math. Verein., **38** (1929), 248–260.
9. T. Kobayashi, *Structures of the Haken manifolds with Heegaard splittings of genus two*, Osaka J. Math. **21** (1984), 437–455.
10. J. H. Lee and S. Lee, *Inequivalent handlebody-knots with homeomorphic complements*, Algebr. Geom. Topol. **12** (2012), 1059–1079.
11. W. B. R. Lickorish, *Prime knots and tangles*, Trans. Amer. Math. Soc. **267** (1981), 321–332.
12. M. Motto, *Inequivalent genus* 2 *handlebodies in* S^3 *with homeomorphic complement*, Topology Appl. **36** (1990), 283–290.
13. H. Schubert. *Die eindeutige Zerlegbarkeit eines Knotens in Primknoten*, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1949 (1949), 57–104.

14. S. Suzuki, *On surfaces in 3-sphere: prime decomposition*, Hokkaido Math. J. **4** (1975), 179–195.

15. Y. Tsukui, *On surfaces in 3-space*, Yokohama Math. J. **18** (1970), 93–104.

16. Y. Tsukui, *On a prime surface of genus 2 and homeomorphic splitting of 3-spheres*, Yokohama Math. J. **23** (1975), 63–75.

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